AN EXAMPLE OF ASYMPTOTICALLY CHOW UNSTABLE MANIFOLDS WITH CONSTANT SCALAR CURVATURE

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Abstract. In [2] Donaldson proved that if a polarized manifold (V, L) has constant scalar curvature Kähler metrics in $c_1(L)$ and its automorphism group $\mathrm{Aut}(V, L)$ is discrete, (V, L) is asymptotically Chow stable. In this paper, we shall show an example which implies that the above result does not hold in the case when $\mathrm{Aut}(V, L)$ is not discrete.

1. Introduction

One of the main issues in Kähler geometry is the existence problem of Kähler metrics with constant scalar curvature on a given Kähler manifold. Through Yau's conjecture [17] and the works of Tian [14], Donaldson [3], this problem is formulated as follows; *The existence of Kähler metrics with constant scalar curvature in a fixed integral Kähler class would be equivalent to a suitable notion of stability of manifolds in the sense of Geometric Invariant Theory.* Though remarkable progress is made recently in this problem, we shall focus only on the related results to our purpose. Let (V, L) be an m-dimensional polarized manifold, that is to say, $L \to V$ is an ample line bundle over an m-dimensional compact complex manifold V. Then the first Chern class $c_1(L)$ of L can be regarded as a Kähler class of V. Let $\mathrm{Aut}(V, L)$ be the group of holomorphic automorphisms of (V, L) modulo the trivial automorphism $\mathbb{C}^\times := \mathbb{C} - \{0\}$. In [2], Donaldson proved that

Theorem 1.1 (Donaldson)**.** *Let (V, L) be a polarized manifold. Assume that* $\mathrm{Aut}(V, L)$ *is discrete. If (V, L) has constant scalar curvature Kähler (cscK) metrics in $c_1(L)$, (V, L) is asymptotically Chow stable.*

The purpose of this paper is to show an example of asymptotically Chow unstable polarized manifolds with cscK metrics in the case of when $\mathrm{Aut}(V, L)$ is not discrete. To state our result more precisely, let us recall the definition of asymptotic Chow stability and some related results. Since L is ample, V can be embedded into the projective space $\mathbb{P}(W) := \mathbb{P}(H^0(V, L^k)^*)$ for sufficiently large k as an algebraic variety $\Psi_{L^k}(V)$. For $\Psi(V)$, there corresponds to a point $[\mathrm{Ch}(\Psi_{L^k}(V))]$ in $\mathbb{P}[\mathrm{Sym}^d(W)^{\otimes(m+1)}]$, which is often called the Chow point (cf. see [12] for the full detail). Take an element $\mathrm{Ch}(\Psi_{L^k}(V))$ representing the Chow point $[\mathrm{Ch}(\Psi_{L^k}(V))]$. The action of the special linear group $\mathrm{SL}(W, \mathbb{C})$ on W is extended to the action on $\mathrm{Sym}^d(W)^{\otimes(m+1)}$. We call $\Psi_{L^k}(V)$ Chow stable if and only if the orbit $\mathrm{SL}(W, \mathbb{C}) \cdot \mathrm{Ch}(\Psi_{L^k}(V))$ is closed and its stabilizer is finite. We call it Chow semistable if and only if the closure of the orbit does not contain the origin. Also we call (V, L) asymptotically Chow (semi-)stable if and only if $\Psi_{L^k}(V)$ is Chow (semi-)stable for all sufficiently large k. In this paper, we say that (V, L) is asymptotically Chow unstable if (V, L) is

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not asymptotically Chow semistable. Theorem 1.1 is extended by Mabuchi [10] to the case when $\mathrm{Aut}(V,L)$ is not discrete.

Theorem 1.2 (Mabuchi)**.** *Let (V,L) be a polarized manifold. Assume that the obstruction introduced in* [9] *vanishes. If (V,L) has cscK metrics in $c_1(L)$, (V,L) is asymptotically Chow polystable in the sense of* [10].

The notion of polystability in the above is defined by that the orbit of $\Psi_{L^k}(V)$ with respect to the action of $\mathrm{SL}(W,\mathbb{C})$ is closed and some additional condition about the isotropy group of $\mathrm{SL}(W,\mathbb{C})$ at $\Psi_{L^k}(V)$. So polystability implies semistability. The obstruction in the above is defined in [9] as a necessary condition for (V,L) to be asymptotically Chow semistable. This obstruction is reformulated by Futaki [6] in more general form by generalizing so-called Futaki invariant. The original Futaki invariant [4] is a map $f:\mathfrak{h}(V)\to\mathbb{C}$ defined by

$$f(X):=\int_V Xh_\omega\omega^m,$$

where $\mathfrak{h}(V)$ is the Lie algebra of all holomorphic vector fields on V, ω is a Kähler form and h_ω is a real-valued function defined by

$$s(\omega)-\left(\int_V s(\omega)\omega^m\Big/\int_V\omega^m\right)=-\Delta_\omega h_\omega$$

up to addition of a constant. Here $s(\omega)$ denotes the scalar curvature of ω, $\Delta_\omega:=-g^{i\bar{j}}\partial_i\bar{\partial}_j$ denotes the complex Laplacian with respect to ω, and $(g^{i\bar{j}})_{i\bar{j}}$ denotes the inverse of $(g_{i\bar{j}})_{i\bar{j}}$. It is well-known that f is independent of the choices of ω and that the vanishing of f is an obstruction for the existence of cscK metrics in the Kähler class $[\omega]$.

Now let us recall the definition of Futaki's obstruction for asymptotic Chow semistability. Let $\mathfrak{h}_0(V)$ be the Lie subalgebra of $\mathfrak{h}(V)$ consisting of holomorphic vector fields which have non-empty zero set. For any $X\in\mathfrak{h}_0(V)$, there exists a complex valued smooth function u_X such that

$$i(X)\omega=-\overline{\partial}u_X,$$

$$\int_V u_X\,\omega^m=0. \tag{1}$$

Let θ be a type $(1,0)$ connection of the holomorphic tangent bundle $T'V$. Let $\Theta:=\bar{\partial}\theta$, which is the curvature form with respect to θ. For $X\in\mathfrak{h}(V)$, let $L(X):=\nabla_X-\mathcal{L}_X$, where ∇_X and $\mathcal{L}_X$ are the covariant derivative by X with respect to θ and the Lie derivative respectively. Remark that $L(X)$ can be regarded as a smooth section of $\mathrm{End}(T'V)$ the endomorphism bundle of the holomorphic tangent bundle. Let ϕ be a $GL(m,\mathbb{C})$-invariant polynomial of degree p on $\mathfrak{gl}(m,\mathbb{C})$. We define $\mathcal{F}_\phi:\mathfrak{h}_0(V)\to\mathbb{C}$ by

$$\begin{aligned}\mathcal{F}_\phi(X) &= (m-p+1)\int_V\phi(\Theta)\wedge u_X\,\omega^{m-p}\\ &\quad+\int_V\phi(L(X)+\Theta)\wedge\omega^{m-p+1}.\end{aligned} \tag{2}$$

It is proved that $\mathcal{F}_\phi(X)$ is independent of the choices of ω and θ (see [6]). Let Td^p be the p-th Todd polynomial which is a $GL(m,\mathbb{C})$-invariant polynomial of degree p on $\mathfrak{gl}(m,\mathbb{C})$. Then it is proved [6]

Theorem 1.3 (Futaki)**.** *If (V,L) is asymptotically Chow semistable, then $\mathcal{F}_{\mathrm{Td}^p}$ vanishes for any $p=1,\cdots,m$.*

In particular $\mathcal{F}_{\mathrm{Td}^1}$ equals to $f\mid_{\mathfrak{h}_0(V)}$ up to multiplication of a constant. The vanishing of $\mathcal{F}_{\mathrm{Td}^p}$ for all p is equivalent to the vanishing of Mabuchi's obstruction (cf. Proposition 4.1 in [6]). In [8], Futaki and the first and second authors investigated the linear dependence among $\{\mathcal{F}_{\mathrm{Td}^p}\}_p$ and proposed the following question.

Problem 1.4. *Does the existence of cscK metrics induce the vanishing of $\mathcal{F}_{\mathrm{Td}^p}$ for all p?*

When $p = 1$, the existence of cscK metrics of course implies the vanishing of $\mathcal{F}_{\mathrm{Td}^1}$. If the answer is affirmative, the assumption of Theorem 1.2 could be omitted and Theorem 1.1 could be extended to the case when $\mathrm{Aut}(V, L)$ is not discrete. They also claim that if a counterexample to Problem 1.4 exists among toric Fano manifolds with anticanonical polarization, it should be a non-symmetric toric Fano manifold with Kähler-Einstein metrics in the sense of Batyrev-Selivanova [1]. At the point when [8] is written, the existence of such toric Fano manifolds was not known. However it is discovered by Nill-Paffenholz [13] very recently. The main result of this paper is to show that one of the toric Fano manifolds in [13] is the desired example in [8]. That is to say,

Theorem 1.5. *There exists a 7-dimensional toric Fano manifold V with Kähler-Einstein metrics in $c_1(V) := c_1(K_V^{-1})$, whose $\mathcal{F}_{\mathrm{Td}^p}$ does not vanish for $2 \leq p \leq 7$. In particular, even if an anti-canonical polarized manifold admits Kähler-Einstein metrics, it is not necessarily asymptotically Chow semistable.*

Also Theorem 1.5 implies that the assumption about obstruction in Theorem 1.2 can not be omitted. We shall prove Theorem 1.5 by the following two ways; the direct calculation by the localization formula (Section 3), and the derivation of the Hilbert series (Section 4). Remark that on Fano manifolds, all cscK metrics in $c_1(V)$ equal to Kähler-Einstein metrics.

2. The Nill-Paffenholz's example

See [13] for notations and terminologies of toric geometry in this section.

First of all, let us recall toric Fano manifolds briefly. A toric variety V is an algebraic normal variety with an effective Hamiltonian action of $T_{\mathbb{C}} := (\mathbb{C}^*)^m$, where $\dim_{\mathbb{C}} V = m$. Let $T_{\mathbb{R}} := (S^1)^m$ be the real torus in $T_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{R}}$ be the associated Lie algebra. Let $N_{\mathbb{R}} := J\mathfrak{t}_{\mathbb{R}} \simeq \mathbb{R}^m$ where J is the complex structure of $T_{\mathbb{C}}$. Let $M_{\mathbb{R}}$ be the dual space $Hom(N_{\mathbb{R}}, \mathbb{R}) \simeq \mathbb{R}^m$ of $N_{\mathbb{R}}$. Denoting the group of algebraic characters of $T_{\mathbb{C}}$ by M, then $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. It is well-known that for each smooth toric Fano manifold V there is a fan Σ_V such that

(a) the polytope Q consisting of the set of the primitive elements of all 1-dimensional cones in Σ_V is an m-dimensional convex polytope,
(b) the origin of $N_{\mathbb{R}}$ is contained in the interior of Q,
(c) any face of Q is a simplex, and
(d) the set of vertices of any $(m-1)$-dimensional face of Q constitutes a basis of $N \simeq \mathbb{Z}^m \subset N_{\mathbb{R}}$.

The polytope Q is often called the Fano polytope of V.

Let V be a seven dimensional toric Fano manifold whose vertices of Fano polytope in $N_{\mathbb{R}}$ are given by

$$
\begin{aligned}
&(\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_5\ \mathbf{v}_6\ \mathbf{v}_7\ \mathbf{v}_8\ \mathbf{v}_9\ \mathbf{v}_{10}\ \mathbf{v}_{11}\ \mathbf{v}_{12}) \\
= &\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 2 & 1 & -1
\end{pmatrix}.
\end{aligned}
\tag{3}
$$

Remark that V is isomorphic to a $\mathbb{P}^1$-bundle over $(\mathbb{P}^1)^3 \times \mathbb{P}^3$.

Theorem 2.1 (Nill-Paffenholz)**.** *The toric Fano manifold V defined by (3) is not symmetric, but its Futaki invariant vanishes. In particular V admits $T_{\mathbb{R}}$-invariant Kähler-Einstein metrics.*

The second statement in Theorem 2.1 follows from the fact proved by Wang-Zhu [15], which says that a toric Fano manifold admits Kähler-Einstein metrics if and only if its Futaki invariant vanishes. Here we shall explain about the symmetry of toric Fano manifolds in Theorem 2.1. Let $\mathrm{Aut}(V)$ be the group of automorphisms of V. Let $\mathcal{W}(V)$ be the Weyl group of $\mathrm{Aut}(V)$ with respect to the maximal torus and $N_{\mathbb{R}}^{\mathcal{W}(V)}$ be the $\mathcal{W}(V)$-invariant subspace of $N_{\mathbb{R}}$. Batyrev and Selivanova [1] says that a toric Fano manifold V is symmetric if and only if $\dim N_{\mathbb{R}}^{\mathcal{W}(V)} = 0$.

Then, let us consider the symmetry of V defined by (3). $\mathcal{W}(V)$ contains a group generated by the following six permutations;

$$
\begin{aligned}
(x_1,x_2,x_3,x_4,x_5,x_6,x_7) &\leftrightarrow (x_2,x_1,x_3,x_4,x_5,x_6,x_7) \\
(x_1,x_2,x_3,x_4,x_5,x_6,x_7) &\leftrightarrow (x_1,x_3,x_2,x_4,x_5,x_6,x_7) \\
(x_1,x_2,x_3,x_4,x_5,x_6,x_7) &\leftrightarrow (x_3,x_2,x_1,x_4,x_5,x_6,x_7) \\
(x_1,x_2,x_3,x_4,x_5,x_6,x_7) &\leftrightarrow (x_1,x_2,x_3,x_5,x_4,x_6,x_7) \\
(x_1,x_2,x_3,x_4,x_5,x_6,x_7) &\leftrightarrow (x_1,x_2,x_3,x_4,x_6,x_5,x_7) \\
(x_1,x_2,x_3,x_4,x_5,x_6,x_7) &\leftrightarrow (x_1,x_2,x_3,x_6,x_5,x_4,x_7),
\end{aligned}
$$

where $(x_1,x_2,x_3,x_4,x_5,x_6,x_7) \in N_{\mathbb{R}} \simeq \mathbb{R}^7$. Hence, we find that the dimension of $N_{\mathbb{R}}^{\mathcal{W}(V)}$ is at most one. However, since V is not symmetric, $\dim N_{\mathbb{R}}^{\mathcal{W}(V)} = 1$.

Next, we shall consider affine toric varieties in V and the associated 7-dimensional cones. The toric Fano manifold V is covered by 64 affine toric varieties, which are isomorphic to $\mathbb{C}^7$ as listed below.

cone	toric affine variety $\simeq \mathbb{C}^7$
$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1, X_2, X_3, Y_1, Y_2, Y_3, Z])$
$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2, X_3, Y_1, Y_2, Y_3, ZX_1^{-1}])$
$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1, X_2, X_3, Y_1Y_3^{-1}, Y_2Y_3^{-1}, Y_3^{-1}, ZY_3^2])$
$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1, X_2, X_3, Y_1, Y_2, Y_3, Z^{-1}])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3, Y_1, Y_2, Y_3, ZX_1^{-1}X_2^{-1}])$
$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2, X_3, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1}, ZX_1^{-1}Y_1^2])$
$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2, X_3, Y_1, Y_2, Y_3, Z^{-1}X_1])$
$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1, X_2, X_3, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1}, Z^{-1}Y_1^{-2})$
$\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3^{-1}, Y_1, Y_2, Y_3, ZX_1^{-1}X_2^{-1}X_3^{-1}])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1},$ $ZX_1^{-1}X_2^{-1}Y_1^2])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3, Y_1, Y_2, Y_3, Z^{-1}X_1X_2])$
$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2, X_3, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1}, Z^{-1}X_1Y_1^{-2}])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3^{-1}, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1},$ $ZX_1^{-1}X_2^{-1}X_3^{-1}Y_1^2])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3^{-1}, Y_1, Y_2, Y_3, Z^{-1}X_1X_2X_3])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1},$ $Z^{-1}X_1X_2Y_1^{-2}])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3^{-1}, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1},$ $Z^{-1}X_1X_2X_3Y_1^{-2}])$

The other affine toric varieties unlisted in the above table can be obtained easily by the symmetry of V.

3. Direct computation of $\mathcal{F}_{\mathrm{Td}^p}$

First, we shall make the family $\{\mathcal{F}_{\mathrm{Td}^p}\}_p$ simpler in the case of the anticanonical polarization. For a Kähler form $\omega \in c_1(L)$, let g be the associated Kähler metric. We have the Levi-Civita connection $\theta = g^{-1}\partial g$ and its curvature form $\Theta = \bar{\partial}\theta$. Then, for the associated covariant derivative ∇ with θ, $L(X)$ can be expressed by

$$L(X) = \nabla X = \nabla_j X^i dz^j \otimes \frac{\partial}{\partial z^i}$$

where $X \in \mathfrak{h}(V)$. Now assume that (V, K_V^{-1}) is a toric Fano manifold with anticanonical polarization. By the Calabi-Yau theorem [16], for a Kähler form $\omega \in c_1(V)$ there exists another Kähler form $\eta \in c_1(V)$ whose Ricci form ρ_η equals to ω. For $X \in \mathfrak{h}_0(V)$ let $\tilde{u}_X$ be the Hamiltonian function with respect to ω and a different normalization from (1)

$$\int_V \tilde{u}_X \omega^m = -f(X).$$

Recall that $\tilde{u}_X = \Delta_\eta \tilde{u}_X$, where Δ_η is the Laplacian of η. Let

$$\begin{aligned} \mathcal{G}_{\mathrm{Td}^p}(X) \quad := \quad & (m-p+1)\int_V \mathrm{Td}^p(\Theta_\eta) \wedge \tilde{u}_X \rho_\eta^{m-p} \\ & + \int_V \mathrm{Td}^p(L_\eta(X) + \Theta_\eta) \wedge \rho_\eta^{m-p+1}. \end{aligned}$$

Here Θ_η is the curvature form of the Levi-Civita connection θ_η with respect to η and $L_\eta(X)$ is also associated with θ_η. The proof of Theorem 3.2 in [8] implies that the difference between $\mathcal{F}_{\mathrm{Td}^p}$ and $\mathcal{G}_{\mathrm{Td}^p}$ equals to a constant multiple of $\mathcal{F}_{\mathrm{Td}^1}$ for any p.

Lemma 3.1. *Let V be a Fano manifold with Kähler-Einstein metrics. Then,*

$$\mathcal{F}_{\mathrm{Td}^p}(X) = \int_V (\mathrm{Td}^p \cdot c_1^{m-p+1})(L_\eta(X) + \Theta_\eta) \tag{4}$$

where $X \in \mathfrak{h}_0(V)$.

Proof. Since V admits Kähler-Einstein metrics and $\mathcal{F}_{\mathrm{Td}^1}$ is proportional to the original Futaki invariant f, $\mathcal{F}_{\mathrm{Td}^1}$ vanishes. So $\mathcal{F}_{\mathrm{Td}^p}$ equals to $\mathcal{G}_{\mathrm{Td}^p}$. Hence, we find

$$\begin{aligned}
\mathcal{F}_{\mathrm{Td}^p}(X) &= (m-p+1)\int_V \mathrm{Td}^p(\Theta_\eta) \wedge (\Delta_\eta u_X)\rho_\eta^{m-p} \\
&\quad + \int_V \mathrm{Td}^p(L_\eta(X) + \Theta_\eta) \wedge \rho_\eta^{m-p+1} \\
&= (m-p+1)\int_V \mathrm{Td}^p(\Theta_\eta) \wedge c_1(L_\eta(X))c_1(\Theta_\eta)^{m-p} \\
&\quad + \int_V \mathrm{Td}^p(L_\eta(X) + \Theta_\eta) \wedge c_1(\Theta_\eta)^{m-p+1} \\
&= \int_V \mathrm{Td}^p(L_\eta(X) + \Theta_\eta) \wedge \\
&\quad \{(m-p+1)c_1(L_\eta(X))c_1(\Theta_\eta)^{m-p} + c_1(\Theta_\eta)^{m-p+1}\} \\
&= \int_V (\mathrm{Td}^p \cdot c_1^{m-p+1})(L_\eta(X) + \Theta_\eta).
\end{aligned}$$

□

Since the right hand of (4) is a kind of the integral invariants in [7], we can apply the localization formula in [7] for $\mathcal{F}_{\mathrm{Td}^p}$ as follows. Assume that X has only isolated zeroes, then

$$\mathcal{F}_{\mathrm{Td}^p}(X) = \sum_{p_i} \frac{\mathrm{Td}^p(L(X)_{p_i})}{\det(L(X)_{p_i})}. \tag{5}$$

where $\{p_i\}$ is the zero set of X. As for the localization formula, see also [5].

Let $\{\sigma_t\}$ be a one-parameter subgroup on V, which induces a holomorphic vector field, defined as follows;

$$\begin{aligned}
&\sigma_t \cdot (X_1, X_2, X_3, Y_1, Y_2, Y_3, Z) \\
= \ &(e^{a_1 t}X_1, e^{a_2 t}X_2, e^{a_2 t}X_3, e^{b_1 t}Y_1, e^{b_2 t}Y_2, e^{b_3 t}Y_3, e^{ct}Z)
\end{aligned}$$

in the affine variety $\mathrm{Spec}(\mathbb{C}[X_1, X_2, X_3, Y_1, Y_2, Y_3, Z])$, which corresponds to the 7-dimensional cone generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$. Remark that

$$X_1 = \frac{x_0}{x_1}, X_2 = \frac{x_2}{x_3}, X_3 = \frac{x_4}{x_5}, Y_1 = \frac{y_0}{y_3}, Y_2 = \frac{y_1}{y_3}, Y_3 = \frac{y_2}{y_3},$$

where

$$([x_0; x_1], [x_2; x_3], [x_4; x_5], [y_0; y_1; y_2; y_3])$$

is the homogeneous coordinates of $(\mathbb{P}^1)^3 \times \mathbb{P}^3$. The action of σ_t on V in another affine variety can be seen from the coordinate transformations (see the table in the previous section). For generic $\{a_i, b_j, c\}_{1 \le i,j \le 3}$, the set of fixed points of σ_t consists of the following isolated 64 points;

$$\{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}, \mathbf{z}) \in V \quad | \quad \mathbf{x}_i, \mathbf{z} = \mathbf{p}_- \text{ or } \mathbf{p}_+, \ \ \mathbf{y} = \mathbf{p}_i \ (i = 1, 2, 3, 4)\},$$

where $\mathbf{p}_-$ denotes $[1; 0]$, $\mathbf{p}_+$ denotes $[0; 1]$ and $\mathbf{p}_1 = [1; 0; 0; 0]$, $\mathbf{p}_2 = [0; 1; 0; 0]$, $\mathbf{p}_3 = [0; 0; 1; 0]$, $\mathbf{p}_4 = [0; 0; 0; 1]$.

Next we shall calculate $L(X)$ at each fixed point of σ_t. For example, let us consider $L(X)$ at

$$(\mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_4, \mathbf{p}_+) = ([0;1], [0;1], [0;1], [0;0;0;1], [0;1]).$$

This point is the origin in an affine variety $\mathrm{Spec}(\mathbb{C}[X_1, X_2, X_3, Y_1, Y_2, Y_3, Z])$ associated with the 7-dimensional cone generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$. Since the holomorphic vector field with respect to σ_t around the point is expressed by

$$\sum_{i=1}^{3} a_i X_i \frac{\partial}{\partial X_i} + \sum_{j=1}^{3} b_i Y_j \frac{\partial}{\partial Y_j} + cZ \frac{\partial}{\partial Z}.$$

Hence $L(X)$ at $(\mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_4, \mathbf{p}_+)$ is given by

$$L(X) = \mathrm{diag}(a_1, a_2, a_3, b_1, b_2, b_3, c).$$

For another example, let us consider $L(X)$ at

$$(\mathbf{p}_-, \mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_1, \mathbf{p}_+) = ([1;0], [0;1], [0;1], [1;0;0;0], [0;1]).$$

This point is the origin in $\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2, X_3, Y_1^{-1}, Y_2 Y_1^{-1}, Y_3 Y_1^{-1}, Z X_1^{-1} Y_1^2])$ associated with a 7-dimensional cone generated by $\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$. Since the holomorphic vector field with respect to σ_t around the point is expressed by

$$-a_1 U_1 \frac{\partial}{\partial U_1} + \sum_{i=2}^{3} a_i U_i \frac{\partial}{\partial U_i} - b_1 U_4 \frac{\partial}{\partial U_4} + \sum_{j=2}^{3} (b_j - b_1) U_{3+j} \frac{\partial}{\partial U_{3+j}} + (c - a_1 + 2b_1) U_7 \frac{\partial}{\partial U_7},$$

where

$$U_1 := X_1^{-1}, U_2 := X_2, U_3 := X_3, U_4 := Y_1^{-1}, U_5 := Y_2 Y_1^{-1}, U_6 := Y_3 Y_1^{-1}, U_7 := Z X_1^{-1} Y_1^2.$$

Hence $L(X)$ at $(\mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_4, \mathbf{p}_+)$ is given by

$$L(X) = \mathrm{diag}(-a_1, a_2, a_3, -b_1, b_2 - b_1, b_3 - b_1, c - a_1 + 2b_1).$$

As for the other fixed points, the computations of $L(X)$ are given by the following table;

no.	fixed pt	$L(X)$
1-1	$(+++,\mathbf{p}_1,\pm)$	$(a_1,a_2,a_3,b_2-b_1,b_3-b_1,-b_1,\pm(c+2b_1))$
1-2	$(-++,\mathbf{p}_1,\pm)$	$(-a_1,a_2,a_3,b_2-b_1,b_3-b_1,-b_1,\pm(c-a_1+2b_1))$
1-3	$(+-+,\mathbf{p}_1,\pm)$	$(a_1,-a_2,a_3,b_2-b_1,b_3-b_1,-b_1,\pm(c-a_2+2b_1))$
1-4	$(++-,\mathbf{p}_1,\pm)$	$(a_1,a_2,-a_3,b_2-b_1,b_3-b_1,-b_1,\pm(c-a_3+2b_1))$
1-5	$(+--,\mathbf{p}_1,\pm)$	$(a_1,-a_2,-a_3,b_2-b_1,b_3-b_1,-b_1,\pm(c-a_2-a_3+2b_1))$
1-6	$(-+-,\mathbf{p}_1,\pm)$	$(-a_1,a_2,-a_3,b_2-b_1,b_3-b_1,-b_1,\pm(c-a_1-a_3+2b_1))$
1-7	$(--+,\mathbf{p}_1,\pm)$	$(-a_1,-a_2,a_3,b_2-b_1,b_3-b_1,-b_1,\pm(c-a_1-a_2+2b_1))$
1-8	$(---,\mathbf{p}_1,\pm)$	$(-a_1,-a_2,-a_3,b_2-b_1,b_3-b_1,-b_1,\pm(c-\sum a_i+2b_1))$
2-1	$(+++,\mathbf{p}_2,\pm)$	$(a_1,a_2,a_3,b_1-b_2,b_3-b_2,-b_2,\pm(c+2b_2))$
2-2	$(-++,\mathbf{p}_2,\pm)$	$(-a_1,a_2,a_3,b_1-b_2,b_3-b_2,-b_2,\pm(c-a_1+2b_2))$
2-3	$(+-+,\mathbf{p}_2,\pm)$	$(a_1,-a_2,a_3,b_1-b_2,b_3-b_2,-b_2,\pm(c-a_2+2b_2))$
2-4	$(++-,\mathbf{p}_2,\pm)$	$(a_1,a_2,-a_3,b_1-b_2,b_3-b_2,-b_2,\pm(c-a_3+2b_2))$
2-5	$(+--,\mathbf{p}_2,\pm)$	$(a_1,-a_2,-a_3,b_1-b_2,b_3-b_2,-b_2,\pm(c-a_2-a_3+2b_2))$
2-6	$(-+-,\mathbf{p}_2,\pm)$	$(-a_1,a_2,-a_3,b_1-b_2,b_3-b_2,-b_2,\pm(c-a_1-a_3+2b_2))$
2-7	$(--+,\mathbf{p}_2,\pm)$	$(-a_1,-a_2,a_3,b_1-b_2,b_3-b_2,-b_2,\pm(c-a_1-a_2+2b_2))$
2-8	$(---,\mathbf{p}_2,\pm)$	$(-a_1,-a_2,-a_3,b_1-b_2,b_3-b_2,-b_2,\pm(c-\sum a_i+2b_2))$
3-1	$(+++,\mathbf{p}_3,\pm)$	$(a_1,a_2,a_3,b_1-b_3,b_2-b_3,-b_3,\pm(c+2b_3))$
3-2	$(-++,\mathbf{p}_3,\pm)$	$(-a_1,a_2,a_3,b_1-b_3,b_2-b_3,-b_3,\pm(c-a_1+2b_3))$
3-3	$(+-+,\mathbf{p}_3,\pm)$	$(a_1,-a_2,a_3,b_1-b_3,b_2-b_3,-b_3,\pm(c-a_2+2b_3))$
3-4	$(++-,\mathbf{p}_3,\pm)$	$(a_1,a_2,-a_3,b_1-b_3,b_2-b_3,-b_3,\pm(c-a_3+2b_3))$
3-5	$(+--,\mathbf{p}_3,\pm)$	$(a_1,-a_2,-a_3,b_1-b_3,b_2-b_3,-b_3,\pm(c-a_2-a_3+2b_3))$
3-6	$(-+-,\mathbf{p}_3,\pm)$	$(-a_1,a_2,-a_3,b_1-b_3,b_2-b_3,-b_3,\pm(c-a_1-a_3+2b_3))$
3-7	$(--+,\mathbf{p}_3,\pm)$	$(-a_1,-a_2,a_3,b_1-b_3,b_2-b_3,-b_3,\pm(c-a_1-a_2+2b_3))$
3-8	$(---,\mathbf{p}_3,\pm)$	$(-a_1,-a_2,-a_3,b_1-b_3,b_2-b_3,-b_3,\pm(c-\sum a_i+2b_3))$
4-1	$(+++,\mathbf{p}_4,\pm)$	$(a_1,a_2,a_3,b_1,b_2,b_3,\pm c)$
4-2	$(-++,\mathbf{p}_4,\pm)$	$(-a_1,a_2,a_3,b_1,b_2,b_3,\pm(c-a_1))$
4-3	$(+-+,\mathbf{p}_4,\pm)$	$(a_1,-a_2,a_3,b_1,b_2,b_3,\pm(c-a_2))$
4-4	$(++-,\mathbf{p}_4,\pm)$	$(a_1,a_2,-a_3,b_1,b_2,b_3,\pm(c-a_3)))$
4-5	$(+--,\mathbf{p}_4,\pm)$	$(a_1,-a_2,-a_3,b_1,b_2,b_3,\pm(c-a_2-a_3))$
4-6	$(-+-,\mathbf{p}_4,\pm)$	$(-a_1,a_2,-a_3,b_1,b_2,b_3,\pm(c-a_1-a_3))$
4-7	$(--+,\mathbf{p}_4,\pm)$	$(-a_1,-a_2,a_3,b_1,b_2,b_3,\pm(c-a_1-a_2))$
4-8	$(---,\mathbf{p}_4,\pm)$	$(-a_1,-a_2,-a_3,b_1,b_2,b_3,\pm(c-\sum a_i))$

As for the notation of the column of fixed points, for example, $(+++,\mathbf{p}_1,-)$ means a fixed point $(\mathbf{p}_+,\mathbf{p}_+,\mathbf{p}_+,\mathbf{p}_1,\mathbf{p}_-)=([0;1],[0;1],[0,1],[1;0;0;0],[1,0])$.

Finally, we shall state the results of calculations of $\mathcal{F}_{\mathrm{Td}^{(p)}}$ $(p=1,\dots,7)$ with respect to the holomorphic vector field induced by $\{\sigma_t\}$ for generic $\{a_i,b_j,c\}_{i,j=1,2,3}$. The following computations are done by the help of a computer algebra system Maxima.[1]

$p=1$**:** In this case we find that $\mathcal{F}_{\mathrm{Td}^1}$ vanishes, because V admits Kähler-Einstein metrics and $\mathcal{F}_{\mathrm{Td}^1}$ is proportional to the ordinary Futaki invariant f. We also can confirm this result by using the localization formula (4) for $\mathcal{F}_{\mathrm{Td}^1}$;

$$2\mathcal{F}_{\mathrm{Td}^1}(X)=\sum_{\mathbf{q}:\ \text{fixed pt}}\frac{c_1^8(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})}=0.$$

[1]Maxima is available from http://maxima.sourceforge.net/.

$p = 2$**:**

$$\begin{aligned} 12\mathcal{F}_{\mathrm{Td}^2}(X) &= \sum_{\mathbf{q}:\ \text{fixed pt}} \frac{(c_1^2 + c_2)c_1^6(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \\ &= \sum_{\mathbf{q}:\ \text{fixed pt}} \frac{(c_2c_1^6)(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \\ &= 13056(\sum a_i - \sum b_i - 2c). \end{aligned}$$

$p = 3$**:**

$$\begin{aligned} 24\mathcal{F}_{\mathrm{Td}^3}(X) &= \sum_{\mathbf{q}:\ \text{fixed pt}} \frac{c_2c_1^6(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \\ &= 12\mathcal{F}_{\mathrm{Td}^2}(X) \\ &= 13056(\sum a_i - \sum b_i - 2c). \end{aligned}$$

$p = 4$**:**

$$\begin{aligned} 720\mathcal{F}_{\mathrm{Td}^4}(X) &= \sum_{\mathbf{q}:\ \text{fixed pt}} \frac{(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4)c_1^4(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \\ &= 94080(\sum a_i - \sum b_i - 2c). \end{aligned}$$

$p = 5$**:**

$$\begin{aligned} 1440\mathcal{F}_{\mathrm{Td}^5}(X) &= \sum_{\mathbf{q}:\ \text{fixed pt}} \frac{(-c_1^3c_2 + 3c_1c_2^2 + c_1^2c_3 - c_1c_4)c_1^3(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \\ &= 28800(\sum a_i - \sum b_i - 2c). \end{aligned}$$

$p = 6$**:**

$$\begin{aligned} 60480\mathcal{F}_{\mathrm{Td}^6}(X) &= \sum_{\mathbf{q}:\ \text{fixed pt}} \Bigg(\frac{(2c_1^6 - 12c_1^4c_2 + 11c_1^2c_2^2 + 10c_3^2 + 5c_1^3c_3)c_1^2(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \\ &\qquad + \frac{(11c_1c_2c_3 - c_3^2 - 5c_1^2c_4 - 9c_2c_4 - 2c_1c_5 + 2c_6)c_1^2(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \Bigg) \\ &= 82176(\sum a_i - \sum b_i - 2c). \end{aligned}$$

$p = 7$**:**

$$\begin{aligned} 120960\mathcal{F}_{\mathrm{Td}^7}(X) &= \sum_{\mathbf{q}:\ \text{fixed pt}} \Bigg(\frac{(11c_1^2c_2c_3 - 9c_1c_2c_4 + 2c_1c_6 - 2c_1^2c_5)c_1(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \\ &\qquad + \frac{(2c_1^3c_4 - c_1c_3^2 - 2c_1^4c_3 + 10c_1c_2^3 - 10c_1^3c_2^2 + 2c_1^5c_2)c_1(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \Bigg) \\ &= 16128(\sum a_i - \sum b_i - 2c). \end{aligned}$$

Remark that all $\mathcal{F}_{\mathrm{Td}^p}$ $(2 \leq p \leq 7)$ are proportional to each other. This result is consistent with the fact that $\dim N_{\mathbb{R}}^{\mathcal{W}(V)} = 1$. Therefore we can conclude that even if a Fano manifold admits Kähler-Einstein metrics (i.e., constant scalar curvature metrics), $\{\mathcal{F}_{\mathrm{Td}^p}\}_{p=1,\dots,m}$ may not vanish. The proof of the main theorem is completed.

4. The derivation of the Hilbert series

In [8], Futaki and the first two authors showed a relation between the obstructions to asymptotic Chow semistability and the derivatives of the Hilbert series. In the present section, we will see that we can also show Theorem 1.5 using such relation.

We first review the definition and some properties of the Hilbert series. See [8] for more detail. Let V be a toric Fano m-fold and Q be the corresponding Fano polytope. The polar dual P of Q, which is the Delzant polytope of V in $M_{\mathbb{R}} \simeq \mathbb{R}^m$, is defined as

$$P := \{w \in \mathbb{R}^m \,|\langle \mathbf{v}_j, w\rangle \geq -1\}$$

where $\mathbf{v}_j \in \mathbb{Z}^m$ is a vertex of Q for each j.

We call the convex rational polyhedral cone

$$\mathcal{C}^* := \{y \in \mathbb{R}^{m+1} \,|\langle\ \lambda_j, y\rangle \geq 0\}$$

the toric diagram of V, where $\lambda_j = (\mathbf{v}_j, 1) \in \mathbb{Z}^{m+1}$. Note here that this cone is a pointed cone in $\mathbb{R}^{m+1}$, that is to say, $\mathcal{C}^* \cap (-\mathcal{C}^*) = \{0\}$. We can also represent $\mathcal{C}^*$ by

$$\mathcal{C}^* = \{c_1\boldsymbol{\mu}_1 + \cdots + c_{m+1}\boldsymbol{\mu}_{m+1} \,|\, c_i \geq 0,\ i = 1, \cdots, m+1\}$$

where $\boldsymbol{\mu}_j = (\mathbf{w}_j, 1) \in \mathbb{Z}^{m+1}$ and $\mathbf{w}_1, \cdots, \mathbf{w}_k$ are the vertices of the Delzant polytope P. Then we can define the (multi-graded) Hilbert series $C(\mathbf{x}, \mathcal{C}^*)$ of the rational cone $\mathcal{C}^*$ by

$$C(\mathbf{x}, \mathcal{C}^*) = \sum_{\mathbf{a} \in \mathcal{C}^* \cap \mathbb{Z}^{m+1}} \mathbf{x}^{\mathbf{a}} \qquad (\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_{m+1}^{a_{m+1}}).$$

As proved in [11], the Hilbert series $C(\mathbf{x}, \mathcal{C}^*)$ can be written a rational generating function of the form

$$C(\mathbf{x}, \mathcal{C}^*) = \frac{K_{\mathcal{C}^*}(\mathbf{x})}{(1 - \mathbf{x}^{\boldsymbol{\mu}_1}) \cdots (1 - \mathbf{x}^{\boldsymbol{\mu}_k})}$$

where $K_{\mathcal{C}^*}(\mathbf{x})$ is a Laurent polynomial. Using Brion's formula, we are able to calculate the right hand side of the above equation as follows, see [8];

$$C(\mathbf{x}, \mathcal{C}^*) = \sum_{j=1}^{k} \frac{1}{1 - \mathbf{x}^{\boldsymbol{\mu}_j}} \prod_{b=1}^{m} \frac{1}{(1 - \tilde{\mathbf{x}}^{\mathbf{e}_{j,b}})},$$

where $\mathbf{e}_{j,1}, \cdots, \mathbf{e}_{j,m} \in \mathbb{Z}^m$ denote the generators of the edges emanating from a vertex $\mathbf{w}_j$ and $\tilde{\mathbf{x}} = (x_1, \cdots, x_m)$.

Let $\mathcal{C}_R$ be the convex polytope defined as

$$\mathcal{C}_R = \{\xi \in \mathcal{C} \,|\, \xi = (\mathbf{b}, m+1)\},$$

where $\mathcal{C}$ is the interior of the dual cone of $\mathcal{C}^*$. For $\xi = (\mathbf{b}, m+1) \in \mathcal{C}_R$ we write

$$\mathbf{e}^{-t\xi} = (e^{-b_1 t}, \cdots, e^{-b_m t}, e^{-(m+1)t})$$

and consider

$$C(\mathbf{e}^{-t\xi}, \mathcal{C}^*) = \frac{K_{\mathcal{C}^*}(\mathbf{e}^{-t\xi})}{(1 - e^{-t\langle \boldsymbol{\mu}_1, \xi\rangle}) \cdots (1 - e^{-t\langle \boldsymbol{\mu}_k, \xi\rangle})}.$$

For each fixed $\xi \in \mathcal{C}_R$, the Laurent expansion of $C(\mathbf{e}^{-t\xi}, \mathcal{C}^*)$ at $t = 0$ is written as

$$C(\mathbf{e}^{-t\xi}, \mathcal{C}^*) = \frac{C_{-m-1}(\mathbf{b})}{t^{m+1}} + \frac{C_{-m}(\mathbf{b})}{t^m} + \frac{C_{-m+1}(\mathbf{b})}{t^{m-1}} + \cdots.$$

In [8], it was showed that the following relation between the invariants $\mathcal{F}_{\mathrm{Td}^p}$ and the derivatives of $C_i(\mathbf{b})$ at $\mathbf{b} = \mathbf{0}$.

Theorem 4.1. *The linear span of the derivatives $d_0C_i(\mathbf{b})$, $i = -m-1, -m, \dots,$ coincides with the linear span of $\mathcal{F}_{\mathrm{Td}^1}, \dots \mathcal{F}_{\mathrm{Td}^m}$ restricted to $\mathfrak{t} \otimes \mathbb{C} \simeq \mathbb{C}^m$.*

Therefore, in principle, if we determine all the data $\{\mathbf{w}_j, \mathbf{e}_{j,b}\}$ from the Fano polytope P, we see whether all $\mathcal{F}_{\mathrm{Td}^p}$ vanish or not. Unfortunately, however, it is difficult to check it directly (even by using computer softwares), because in our case the Hilbert series has so many terms and each one is complicated.

To solve this problem, we use the following proposition.

Proposition 4.2. *If $\mathcal{F}_{\mathrm{Td}^p} = 0$ for each $p = 1, \dots, m$ then*

$$\frac{\partial}{\partial x} C(x^{n_1}, \dots, x^{n_m}, e^{-(m+1)t})_{|x=1} = 0 \tag{6}$$

for any $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m$.

Proof. If $\mathcal{F}_{\mathrm{Td}^p} = 0$ for each $p = 1, \dots, m$ then

$$\frac{\partial}{\partial b_i} C(e^{-b_1 t}, \dots, e^{-b_m t}, e^{-(m+1)t})_{|\mathbf{b}=0} = 0$$

holds for each $i = 1, \dots, m$ by Theorem 4.1. Hence we easily see the proposition by the chain rule. $\square$

The left hand side of (6) for Nill-Paffenholz's is computable with computer. The combinatorial data we need is in Appendix. For example, in consequence of the Maple calculation, we can see

$$\frac{\partial}{\partial x} C(x^{n_1}, \cdots, x^{n_7}, e^{-8t})_{|x=1} = -\frac{184e^{-8t}(2e^{-32t} + 31e^{-24t} + 70e^{-16t} + 31e^{-8t} + 2)}{(-1 + e^{-8t})^7} \neq 0$$

for $(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (1, 2, 3, 4, 5, 6, 7)$.

Thus, by Proposition 4.2, Theorem 1.5 has been proved.

5. Appendix: combinatorial data of Nill-Paffenholz's example

In this section we shall list up the necessary combinatorial data of Nill-Paffenholz's example for the calculation in Section 4.

- The vertices of the Fano polytope Q given by (3),
- The 64 vertices of the polar polytope P given by

$$(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots, \mathbf{w}_{64})$$
$$= \left(\begin{array}{cccccccccccccccc}
-1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 \\
-1 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 \\
-1 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 5 & -1 & -1 & -1 & -1 \\
-1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 5 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1
\end{array}\right.$$

$$\begin{array}{cccccccccccccccccc}
-1 & -1 & 2 & -1 & -1 & 2 & -1 & 2 & -1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & -1 & 0 \\
-1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & 0 & -1 & 2 & 2 & 2 & 2 & 2 & -1 & 2 & 0 \\
2 & -1 & -1 & -1 & 2 & 2 & 0 & -1 & -1 & -1 & 2 & 2 & 2 & 2 & -1 & 2 & 2 & 0 \\
1 & 1 & 1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 5 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1
\end{array}$$

$$
\begin{matrix}
0 & 2 & 2 & -1 & 0 & 2 & 2 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 \\
0 & 2 & -1 & 2 & 0 & 2 & -1 & 2 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 \\
0 & -1 & 2 & 2 & 0 & -1 & 2 & 2 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & 5 \\
-1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 5 & -1 & 5 & 5 & 5 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & 1 & 1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & 5 & 5 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{matrix}
$$

$$
\left.\begin{matrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\
-1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
5 & -1 & -1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & 5 \\
-1 & -1 & -1 & -1 & -1 & 5 & -1 & -1 & 5 & 5 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 5 & 5 & -1 & -1 & 5 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{matrix}\right)
$$

• The neighbors of each vertex of P, given as follows (Here, verteicies v and u of P are called neighbors if the interval $[u, v]$ is an edge of P).

vertex	associated cone	neighbors
$\mathbf{w}_1$	$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5, \mathbf{w}_6, \mathbf{w}_7, \mathbf{w}_8$
$\mathbf{w}_3$	$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4, \mathbf{w}_{13}, \mathbf{w}_{14}, \mathbf{w}_{15}, \mathbf{w}_{16}$
$\mathbf{w}_7$	$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathbf{w}_1, \mathbf{w}_{11}, \mathbf{w}_{15}, \mathbf{w}_{19}, \mathbf{w}_{22}, \mathbf{w}_{24}, \mathbf{w}_{26}$
$\mathbf{w}_8$	$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathbf{w}_1, \mathbf{w}_{12}, \mathbf{w}_{16}, \mathbf{w}_{20}, \mathbf{w}_{23}, \mathbf{w}_{25}, \mathbf{w}_{26}$
$\mathbf{w}_{19}$	$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathbf{w}_4, \mathbf{w}_7, \mathbf{w}_{11}, \mathbf{w}_{15}, \mathbf{w}_{31}, \mathbf{w}_{32}, \mathbf{w}_{53}$
$\mathbf{w}_{20}$	$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathbf{w}_4, \mathbf{w}_8, \mathbf{w}_{12}, \mathbf{w}_{16}, \mathbf{w}_{51}, \mathbf{w}_{52}, \mathbf{w}_{53}$
$\mathbf{w}_{24}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathbf{w}_6, \mathbf{w}_7, \mathbf{w}_{28}, \mathbf{w}_{31}, \mathbf{w}_{36}, \mathbf{w}_{40}, \mathbf{w}_{56}$
$\mathbf{w}_{26}$	$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathbf{w}_7, \mathbf{w}_8, \mathbf{w}_{47}, \mathbf{w}_{50}, \mathbf{w}_{53}, \mathbf{w}_{55}, \mathbf{w}_{56}$
$\mathbf{w}_{27}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathbf{w}_{28}, \mathbf{w}_{29}, \mathbf{w}_{30}, \mathbf{w}_{31}, \mathbf{w}_{32}, \mathbf{w}_{33}, \mathbf{w}_{34}$
$\mathbf{w}_{28}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathbf{w}_{21}, \mathbf{w}_{22}, \mathbf{w}_{24}, \mathbf{w}_{27}, \mathbf{w}_{29}, \mathbf{w}_{30}, \mathbf{w}_{35}$
$\mathbf{w}_{31}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathbf{w}_{18}, \mathbf{w}_{19}, \mathbf{w}_{24}, \mathbf{w}_{27}, \mathbf{w}_{36}, \mathbf{w}_{40}, \mathbf{w}_{44}$
$\mathbf{w}_{34}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathbf{w}_{27}, \mathbf{w}_{35}, \mathbf{w}_{39}, \mathbf{w}_{43}, \mathbf{w}_{44}, \mathbf{w}_{57}, \mathbf{w}_{64}$
$\mathbf{w}_{35}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathbf{w}_{28}, \mathbf{w}_{34}, \mathbf{w}_{39}, \mathbf{w}_{43}, \mathbf{w}_{54}, \mathbf{w}_{55}, \mathbf{w}_{56}$
$\mathbf{w}_{44}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathbf{w}_{31}, \mathbf{w}_{34}, \mathbf{w}_{52}, \mathbf{w}_{53}, \mathbf{w}_{56}, \mathbf{w}_{60}, \mathbf{w}_{61}$
$\mathbf{w}_{53}$	$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathbf{w}_{19}, \mathbf{w}_{20}, \mathbf{w}_{26}, \mathbf{w}_{44}, \mathbf{w}_{47}, \mathbf{w}_{50}, \mathbf{w}_{57}$
$\mathbf{w}_{56}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathbf{w}_{24}, \mathbf{w}_{25}, \mathbf{w}_{26}, \mathbf{w}_{35}, \mathbf{w}_{44}, \mathbf{w}_{60}, \mathbf{w}_{61}$

The other sets of neighbors unlisted in the above table can be obtained by the symmetry of V.

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